Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices

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Abstract

We introduce a family of compatible Poisson brackets on the space of rational functions with denominator of a fixed degree and use it to derive a multi-Hamiltonian structure for a family of integrable lattice equations that includes both the standard and the relativistic Toda lattices.

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1

It has been known since Moser's work on finite non-periodic Toda lattice ([19]) that the space of rational functions plays an important role in solving integrable lattice and constructing action-angle variables. The Moser map that sends the space of $n \times n$ tri-diagonal matrices (the phase space for the Toda lattice) into a space of proper rational functions with a denominator of degree n was later utilized in a more general context (see, e.g. [10], [4], [11]) and generalized in [18] for an arbitrary semisimple Lie algebra to be used, mainly, as a tool for linearization for a class of finite systems of differential-difference equations.

The goal of this paper is to show that the Moser map also serves as an effective tool in establishing a multi-Hamiltonian nature of a class of integrable lattices that includes, in particular, the standard and the relativistic Toda lattices. This class was studied in our recent paper [12]. Our approach is in contrast to those of [8], [9], [22], where the search for compatible Poisson structures was conducted in terms of the matrix entries of the Lax operator. Instead, we introduce explicitly the family of compatible Poisson brackets on the space of rational functions and then pull them back to a phase space of a lattice in question via the Moser map. Note that the linear Poisson structure on the phase space then corresponds to the Atiyah-Hitchin Poisson structure on rational functions [2]. (In the case of the finite non-periodic Toda lattice, this correspondence was first explicitly pointed out in [11].)

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In sect. 2 we review the construction of the family of integrable systems introduced in [12] as restrictions of the full Kostant-Toda flows in $sl(n)$ to elements of a certain family of symplectic leaves. A construction of compatible Poisson structures on rational functions and a consequent description of multi-Hamiltonian structure for lattices of sect. 2 is given in sect. 3. In the last section we give explicit formulae in terms of the corresponding Lax operators for master symmetries that generate these multi-Hamiltonian structure. In the case of the symmetric Toda lattice such master symmetries were implicitly defined via recursive relations in [8].

We conclude this introduction with two natural questions that we would like to address in the future. First, it would be interesting to extend our approach to more general (e.g. generic) symplectic leaves of the full Kostant-Toda flows. Second, we would like to understand how results of this paper can be translated into a geometric language for bi-Hamiltonian systems advocated in [15].

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2

Let us first recall the definition of the Kostant-Toda flows. Denote by e_j $(j = 0, \dots, n)$ vectors of the standard basis in $\mathbb{R}^{n+1}$, by E_{ij} an elementary matrix $(\delta_i^\alpha \delta_j^\beta)_{\alpha,\beta=0}^n$ and by J an $(n+1) \times (n+1)$ matrix with 1s on the first sub-diagonal and 0s everywhere else. Let $\mathfrak{b}_+, \mathfrak{n}_+, \mathfrak{b}_-, \mathfrak{n}_-$ be, resp., algebras of upper triangular, strictly upper triangular, lower triangular and strictly lower triangular $(n+1) \times (n+1)$ matrices. Denote by $\mathcal{H}$ the set $J + \mathfrak{b}_+$ of upper Hessenberg matrices.

For any matrix A we write its decomposition into a sum of lower triangular and strictly upper triangular matrices as

$$A = A_- + A_0 + A_+$$

and define $A_{\geq 0} = A_0 + A_+, A_{\leq 0} = A_0 + A_-$.

The hierarchy of commuting Kostant-Toda flows on $\mathcal{H}$ is given by the family of Lax equations

$$\dot{X} = [X, (X^m)_{\leq 0}] \ (m = 1, \dots, n). \tag{2.1}$$

Each of the flows defined by (2.1) is Hamiltonian with respect to a linear Poisson structure on $\mathcal{H}$ is obtained as a pull-back of the Kirillov-Kostant structure on $\mathfrak{b}_-^*$, the dual of $\mathfrak{b}_-$, if one identifies $\mathfrak{b}_-^*$ and $\mathcal{H}$ via the trace form. Then a Poisson bracket of two functions f_1, f_2 on $\mathcal{H}$ is

$$\{f_1, f_2\}(X) = \langle X, [(\nabla f_1(X))_{\leq 0}, (\nabla f_2(X))_{\leq 0}] \rangle \ , \tag{2.2}$$

where we denote by $\langle X, Y \rangle$ the trace form Trace(XY) and gradients are computed w.r.t. this form. The m-th flow of the hierarchy (2.1) is generated by the Hamiltonian

$$H_{m+1}(X) = \frac{1}{m+1} Tr(X^{m+1}) = \frac{1}{m+1} \sum_{i=0}^{n} \lambda_i^{m+1} \ , \tag{2.3}$$

where $\lambda_i (i = 0, \dots, n)$ are the eigenvalues of X.

The Weyl function

$$M(\lambda) = M(\lambda, X) = ((\lambda \mathbf{1} - X)^{-1} e_0, e_0) = \sum_{j=0}^{\infty} \frac{s_j(X)}{\lambda^{j+1}} \tag{2.4}$$

is an important tool in the study of the Toda flows (2.1) (see, e.g., [4], [7], [10],[11], [19]). Here $s_j(X) = (X^j e_0, e_0)$.

If $X \in \mathcal{H}_0$, where $\mathcal{H}_0 \subset \mathcal{H}$ consists of elements with simple real spectrum $\lambda_1 < \ldots < \lambda_n$, one can write (2.4) as

$$M(\lambda) = \sum_{i=0}^{n} \frac{\rho_i(X)}{\lambda - \lambda_i(X)} \qquad (\sum_{i=0}^{n} \rho_i(X) = 1) \ . \tag{2.5}$$

The Lax equation (2.1) implies the following evolution for $\rho_i(X), \lambda_i(X)$

$$\dot\rho_i(X) = (\lambda_i(X)^m - s_m)\rho_i(X), \ \dot\lambda_i(X) = 0 \ . \tag{2.6}$$

An identity

$$s_m = \sum_{j=0}^{n} \lambda_j^m \rho_j(X) \tag{2.7}$$

allows one to write the system (2.6) in a closed form. The solution of (2.6) is given by

$$\rho_i(X(t)) = \frac{e^{\lambda_i^m t}\rho_i(X(0))}{\sum_{j=0}^{n} e^{\lambda_j^m t}\rho_j(X(0))} \ . \tag{2.8}$$

As is well-known, symplectic leaves of the bracket (2.2) are orbits of the coadjoint action of the group $\mathbf{B}_-$ of lower triangular invertible matrices:

$$\mathfrak{O}_X = \{J + (\mathrm{Ad}_n X)_{\geq 0} \ : n \in \mathbf{B}_-\} \ . \tag{2.9}$$

In [12] we described a family of integrable lattices associated with orbits $\mathfrak{O}_X$ of a special kind. This family contains both the standard and relativistic Toda lattices. Its members are parameterized by increasing sequences of natural numbers $I = \{i_1, \ldots, i_k \ : 0 < i_1 < \ldots < i_k = n\}$. To each sequence I there corresponds a 1-parameter family of $2n$-dimensional coadjoint orbits $M_I = \mathfrak{O}_{X_I + \nu\mathbf{1}} \in \mathcal{H}$, where

$$X_I = e_{0i_1} + \sum_{j=1}^{k-1} e_{i_j i_{j+1}} \ . \tag{2.10}$$

Two Darboux parametrizations for M_I were found in [12]. Each of them allows us to lift the first ($m = 1$) of the Kostant-Toda flows (2.1) on M_I to an integrable flow on $\mathbb{R}^{2n+2}$ equipped with the standard symplectic structure. In one of these parametrizations the Toda Hamiltonian $H_1(X)$ has a form

$$\tilde H_I(Q,P) = \frac{1}{2}\sum_{i=0}^{n} P_i^2 \quad + \sum_{0\leq i<n; i\neq i_1,\ldots,i_{k-1}}^{n} P_i e^{Q_{i+1}-Q_i} \ + \sum_{j=1}^{k-1} e^{Q_{i_j+1}-Q_{i_j}} \ . \tag{2.11}$$

The set M_I' of elements of the form

$$X = (J + D)(\mathbf{1} - C_k)^{-1}(\mathbf{1} - C_{k-1})^{-1} \cdots (\mathbf{1} - C_1)^{-1}, \tag{2.12}$$

where $D = \mathrm{diag}(d_0, \dots, d_n)$

$$C_j = \sum_{\alpha=i_{j-1}}^{i_j-1} c_\alpha e_{\alpha,\alpha+1} \ , \tag{2.13}$$

is dense in M_I.

Then the first ($m = 1$) of the Kostant-Toda flows (2.1) on M'_I is equivalent to the following system

$$\begin{aligned} \dot{d}_i &= d_i(c_i - c_{i-1}), \\ \dot{c}_i &= c_i(d_{i+1} - d_i + c_{i+1} - c_{i-1}) \qquad (i_j < i < i_{j+1}, j = 0, \dots, k-1) \\ \dot{c}_{i_j} &= c_{i_j}\left(d_{i_j+1} - d_{i_j} + (1 - \delta_{i_j+1, i_{j+1}}) c_{i_j+1}\right) \qquad (j = 0, \dots, k-1) \ .) \end{aligned} \tag{2.14}$$

Equations (2.14) can be viewed as particular cases of the *constrained KP lattice* introduced and discretized in [23]. In fact, all the minimal indecomposable invariant submanifolds for the latter system can be obtained this way. To emphasize a connection to Toda flows and the dependence on I we will denote the lattice (2.14) by TL_I. If $I = \{n\}$, we recover the relativistic Toda lattice

$$\dot{c}_i = c_i(d_{i+1} - d_i + c_{i+1} - c_{i-1}), \ \dot{d}_i = d_i(c_i - c_{i-1}) \ . \tag{2.15}$$

If, on the other hand, $I = \{1, 2, \dots, n\}$, then M_I is the set Jac of tri-diagonal matrices in $\mathcal{H}$ with non-zero entries on the super-diagonal and thus, one obtains the standard Toda lattice. In coordinates c_i, d_i equations of motion form a system

$$\dot{d}_i = d_i(c_i - c_{i-1}), \ \dot{c}_i = c_i(d_{i+1} - d_i), \tag{2.16}$$

which, after relabeling $d_i = u_{2i-1}, c_i = u_{2i}$, becomes the Volterra lattice

$$\dot{u}_i = u_i(u_{i+1} - u_{i-1}).$$

In [12] we proved the following

Proposition 2.1 *For any I, there exists a unique birational transformation of the from $X \to Ad_{n(X)}X$ from M_I to Jac, that preserves the Weyl function $M(\lambda, X)$ and, for $k = 1, \dots, n$, sends the k-th Toda flow (2.1) on M_I into the k-th Toda flow on Jac. Here $n(X)$ is a unipotent upper triangular matrix, whose off-diagonal elements in the first row are all equal to zero.*

One of the consequences of Proposition 2.1 is that, for any I, on the open dense set in M_I, an element $X \in M_I$ can be uniquely determined by its Weyl function $M(\lambda, X)$. (This, of course, is well-known in the case of the tri-diagonal and relativistic Toda lattices, see, e.g., [19], [4], [7], [17].) In the next section, we use this fact to derive the multi-Hamiltonian structure for systems TL_I.

3

Let Rat_{n+1} denote a space of rational functions of the form

$$m(\lambda) = \frac{q(\lambda)}{p(\lambda)} = \sum_{i=0}^{\infty} \frac{h_i}{\lambda^{i+1}} \ , \tag{3.1}$$

where $p(\lambda)$ is a monic polynomial of degree $n+1$ and $q(\lambda)$ is a polynomial of degree less than $n+1$. To define a Poisson bracket on Rat_{n+1}, it is sufficient to specify pairwise brackets for $p(\lambda), q(\lambda), p(\mu), q(\mu)$, where λ and μ are arbitrary.

For fixed $p(\lambda), q(\lambda)$ and $k = 0, \dots, n$, let us denote

$$q^{[k]}(\lambda) = \lambda^k q(\lambda) \pmod{p(\lambda)} \tag{3.2}$$

and define a skew-symmetric bracket $\{\ ,\ \}_k$ on coefficients of $p(\lambda), q(\lambda)$ by setting

$$\{p(\lambda), p(\mu)\}_k = \{q(\lambda), q(\mu)\}_k = 0 \tag{3.3}$$

and

$$\{p(\lambda), q(\mu)\}_k = \frac{p(\lambda)q^{[k]}(\mu) - p(\mu)q^{[k]}(\lambda)}{\lambda - \mu}\ . \tag{3.4}$$

Proposition 3.1 *$\{\ ,\ \}_k$ ($k = 0, \dots, n$) are compatible Poisson structures on Rat_{n+1}.*

Proof. It is sufficient to check the statement on an open dense subset of Rat_{n+1} defined by the assumption that $p(\lambda)$ and $q(\lambda)$ are co-prime and all roots $\lambda_0, \dots \lambda_n$ of $p(\lambda)$ are distinct. On this subset

$$m(\lambda) = \frac{q(\lambda)}{p(\lambda)} = \sum_{i=0}^{n} \frac{r_i}{\lambda - \lambda_i}$$

and the data $\{\lambda_i, q(\lambda_i), i = 0, \dots, n\}$ determines $p(\lambda)$ and $q(\lambda)$ completely. In particular,

$$r_k = q(\lambda_i) \prod_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{q(\lambda_i)}{p'(\lambda_i)} \tag{3.5}$$

By (3.3),

$$\{\lambda_i, \lambda_j\}_k = 0 \tag{3.6}$$

Since

$$0 = \{p(\lambda_i), q(\mu)\}_k = p'(\lambda_i)\{\lambda_i, q(\mu)\}_k + \{p(\lambda), q(\mu)\}_k \restriction_{\lambda=\lambda_i},$$

one obtains from (3.4)

$$\{\lambda_i, q(\mu)\}_k = \frac{p(\mu)q^{[k]}(\lambda_i)}{p'(\lambda_i)(\lambda_i - \mu)} = \frac{p(\mu)\lambda_i^k q(\lambda_i)}{p'(\lambda_i)(\lambda_i - \mu)} = -\lambda_i^k q(\lambda_i) \prod_{j \neq i} \frac{\mu - \lambda_j}{\lambda_i - \lambda_j}\ , \tag{3.7}$$

which, together with (3.6), implies

$$\{\lambda_i, q(\lambda_j)\}_k = -\lambda_i^k q(\lambda_i)\delta_i^j, \tag{3.8}$$

and, consequently,

$$\{q(\lambda_i), q(\lambda_j)\}_k = 0\ . \tag{3.9}$$

It follows from (3.6), (3.8), (3.9), that in coordinates $\lambda_i, q_i = q(\lambda_i)$, any linear combination $\{\ ,\ \}_c = \sum_{k=0}^n c_k \{\ ,\ \}_k$ has a form

$$\begin{aligned} \{\lambda_i, \lambda_j\}_c &= \{q_i, q_j\}_c = 0\ , \\ \{\lambda_i, q_j\}_c &= -c(\lambda_i) q_i \delta_i^j\ , \end{aligned} \tag{3.10}$$

where $c(\lambda) = \sum_{k=0}^n c_k \lambda^k$. It is easy to see that the bracket defined by (3.10) satisfies the Jacobi identity, with canonical coordinates given by

$$x_i = \int \frac{\mathrm{d}\lambda_i}{c(\lambda_i)}\ ,\ y_i = \ln q_i\ (i = 0, \dots, n). \tag{3.11}$$

Thus, any linear combination of $\{\ ,\ \}_k$ is a Poisson bracket, which finishes the proof. $\square$

Remarks. 1. The expression in the right hand side of (3.4) is called *a Bezoutian* of polynomials $p(\lambda)$ and $q^{[k]}(\lambda)$. For more information on bezoutians and the role they in the control theory we refer the reader to a survey [14]

2. When $k = 0$, brackets (3.3), (3.4) give Atiyah-Hitchin Poisson structure on Rat_{n+1} [2].

Poisson structure (3.3), (3.4) can be re-written directly in terms of the elements $m(\lambda) \in Rat_{n+1}$ as follows

$$\{m(\lambda), m(\mu)\}_k = \Big((\lambda^k m(\lambda))_- - (\mu^k m(\mu))_- \Big) \frac{m(\lambda) - m(\mu)}{\lambda - \mu}\ , \tag{3.12}$$

where, for a rational function $r(\lambda)$, $(r(\lambda))_+$ denotes the polynomial part of its Laurent decomposition and $r(\lambda)_- = r(\lambda) - (r(\lambda))_+$. It follows from (3.12) that , in terms of coefficients h_i of the Laurent expansion (3.1) of $m(\lambda)$, $\{\ ,\ \}_k$ has a form

$$\{h_i, h_j\}_k = \sum_{\alpha=i}^{j} h_{k+\alpha} h_{i+j-1-\alpha}\quad (i < j) \tag{3.13}$$

Now we can restrict brackets $\{\ ,\ \}_k$ to a subset of Rat_{n+1} that contains the image of the map $M(\lambda, \cdot) : \mathcal{H} \to Rat_{n+1}$ described by (2.4). This subset, denoted by Rat'_{n+1} is the set of all $M(\lambda) = \frac{q(\lambda)}{p(\lambda)} \in Rat_{n+1}$ with both polynomials $q(\lambda)$ and $p(\lambda)$ monic. To compute a Poisson bracket induced by $\{\ ,\ \}_k$ on Rat'_{n+1}, we first note that, by (3.4), the bracket between h_0, the leading coefficient of $q(\lambda)$, and $p(\lambda)$ is $\{p(\lambda),\ h_0\}_k = q^{[k]}(\lambda)$. Then, since $M(\lambda) = \frac{1}{h_0} m(\lambda)$ defines a surjective map from Rat_{n+1} to Rat'_{n+1}, a straightforward computation leads to the following

Proposition 3.2 *The family of compatible Poisson structures on Rat'_{n+1} is given by*

$$\{M(\lambda), M(\mu)\}_k = \Big((\lambda^k M(\lambda))_- - (\mu^k M(\mu))_- \Big) \left(\frac{M(\lambda) - M(\mu)}{\lambda - \mu} + M(\lambda) M(\mu) \right)\ . \tag{3.14}$$

Any linear combination $\{\ ,\ \}_c = \sum_{k=0}^n c_k \{\ ,\ \}_k$ of brackets (3.14) is degenerate. Indeed, it follows from (3.7), that

$$\{\lambda_i,\ h_0\}_k = -\frac{\lambda_i^k q_i}{p'(\lambda_i)}\ . \tag{3.15}$$

Let $F = \sum_{i=0}^{n} x_i$, where x_i are defined in (3.11). Then, clearly, $\{p(\lambda), F\}_c = 0$ and

$$\{\frac{q(\lambda)}{h_0}, F\}_c = \sum_{i=0}^{n} q(\lambda_i) \prod_{j\neq i} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j} - \frac{q(\lambda)}{h_0} \sum_{i=0}^{n} \frac{q(\lambda_i)}{h_0 p'(\lambda_i)}$$

But, by (3.5), $\sum_{i=0}^{n} \frac{q(\lambda_i)}{h_0 p'(\lambda_i)} = 1$ and so $\{\frac{q(\lambda)}{h_0}, F\}_c = 0$ by the Lagrange interpolation formula. Thus, $F = \sum_{i=0}^{n} x_i$ is a Casimir for $\{\ ,\ \}_c$ on Rat'_{n+1}, while canonical coordinates coordinates for this bracket can be easily derived from (3.11):

$$x_i = \int \frac{d\lambda_i}{c(\lambda_i)}\ ,\ y_i = \ln \frac{q_i}{q_n}\ (i = 0, \ldots, n-1). \tag{3.16}$$

Denote $\rho_i = \frac{r_i}{h_0} = \frac{q_i}{h_0 p'(\lambda_i)}$ and define $H_j = \frac{1}{j} \sum_{l=0}^{n} \lambda_l^j (j = \pm 1, \pm 2, \ldots)$ and $H_0 = \sum_{l=0}^{n} \ln \lambda_l$. Then (3.8), (3.15) imply

$$\{\rho_i, H_j\}_k = (\lambda_i^{k+j-1} - \sum_{l=0}^{n} \lambda_l^{k+j-1} \rho_l) \rho_i\ .$$

Comparing the last equation with (2.6), (2.7), one sees that equations of motion induced on Rat'_{n+1} via the map (2.4) by the m-th Toda flow (2.1) coincide with Hamilton equations generated in the Poisson structure $\{\ ,\ \}_k$ by the Hamiltonian H_{m+1-k}. Now, by Proposition 2.1, for any index set I, (2.4) defines an almost everywhere invertible map from M_I to Rat'_{n+1}. Moreover, the bracket (3.14) is polynomial in terms of the coefficients of the Laurent expansion $M(\lambda, X) = \sum_0^\infty h_j \lambda^{-j-1} = \sum_0^\infty (L^j e_0, e_0) \lambda^{-j-1}$, the well-known determinantal formulae, expressing the entries of the element of Jac via h_j are rational in h_j ([1]), and, by Proposition 2.1, so are formulae expressing matrix entries of elements of M_I in terms of h_j. Thus each of the brackets in (3.14) uniquely defines a Poisson bracket on M_I which is rational if written in terms of either matrix entries of elements of M_I or in terms of the parameters c_i, d_i defined on M'_I by (2.12), (2.13). We obtain

Theorem 3.3 *For any I, the restriction of the hierarchy (2.1) to $M_I \subset \mathcal{H}$ possesses a multi-Hamiltonian structure. Compatible Poisson brackets $\{\ ,\ \}_k^I$ $(k = 0, \ldots, n)$ for this structure are obtained as a pull-back of the Poisson brackets (3.14) via the restriction of the map (2.4) to M_I. The m-th flow of the hierarchy (2.1) is generated in the Poisson structure $\{\ ,\ \}_k^I$ by the Hamiltonian $H_{m+1-k}(X)$, where*

$$H_j = \begin{cases} \frac{1}{j} Tr(X^j) & j \neq 0 \\ \ln \det(X) & j = 0 \end{cases}\ . \tag{3.17}$$

Theorem 3.3 provides a uniform way of constructing a multi-Hamiltonian structure for both standard and relativistic Toda lattices, as well as all lattices TL_I. Furthermore, since the relativistic Toda hierarchy is connected to the Ablowitz-Ladik one ([3]) via a birational transformation (cf., e.g. [12], [17]), one can use Theorem 3.3 to construct a multi-Hamiltonian structure for the latter hierarchy too. Note also, that for any I, $\{\ ,\ \}_0^I$ is the restriction of the bracket (2.2) to M_I

We shall now derive a formula that, for fixed λ and μ, expresses $\{M(\lambda, X), M(\mu, X)\}_k$ in terms of X, that agrees with the formula conjectured (and proved for $k = 0, 1, 2$) in [8] for compatible Poisson brackets for the symmetric Toda lattice. First, recall the definition of the R-matrix associated with the Lax equation 2.1. It is defined by $R(A) = (A)_{\leq 0} - (A)_{>0}$ (see, e.g. [20]).

Proposition 3.4

$$\{M(\lambda,X),M(\mu,X)\}_k = \frac{1}{4}\left\langle X, \left[R(X^k\nabla_\lambda+\nabla_\lambda X^k),\nabla_\mu\right] + \left[\nabla_\lambda, R(X^k\nabla_\mu+\nabla_\mu X^k)\right]\right\rangle \ , \qquad (3.18)$$

where $\nabla_\lambda = \nabla M(\lambda, X)$.

Proof. Denote $R_\lambda = (\lambda\mathbf{1} - X)^{-1}$. It follows from (2.4) that $\nabla_\lambda = \nabla M(\lambda, X) = R_\lambda E_{00} R_\lambda$. The following identities are easily checked:

$$\frac{1}{\lambda-\mu}(R_\lambda - R_\mu) = -R_\lambda R_\mu \ , \quad [X,\nabla_\lambda] = [R_\lambda, E_{00}] \ . \qquad (3.19)$$

Note also that $(\lambda^k M(\lambda))_- = (X^k R_\lambda e_0, e_0)$. Taking (3.19) into an account, one sees that the second factor in the right-hand side of (3.14) is equal to $(R_\lambda e_0, e_0)(R_\mu e_0, e_0) - (R_\lambda R_\mu e_0, e_0) = e_0^T R_\lambda(E_{00} - \mathbf{1})R_\mu e_0$,while the first factor is equal to $((X^k(R_\lambda - R_\mu)e_0, e_0) = e_0^T X^k(R_\lambda - R_\mu)e_0$.

Thus,

$$\begin{aligned}\{M(\lambda,X),M(\mu,X)\}_k &= Tr\left(X^k(R_\lambda - R_\mu)E_{00}R_\lambda(E_{00}-\mathbf{1})R_\mu E_{00}\right)\\ &= Tr\left(X^k\nabla_\lambda(E_{00}-\mathbf{1})R_\mu E_{00} - E_{00}R_\lambda(E_{00}-\mathbf{1})\nabla_\mu X^k\right)\\ &= Tr\left(X^k\nabla_\lambda\left[E_{00},R_\mu\right]E_{00} - E_{00}\left[E_{00},R_\lambda\right]\nabla_\mu X^k\right)\\ &\overset{(3.19)}{=} Tr\left(E_{00}X^k\nabla_\lambda\left[\nabla_\mu, X\right] - \left[\nabla_\lambda, X\right]\nabla_\mu X^k E_{00}\right)\end{aligned} \qquad (3.20)$$

Since the Weyl function $M(\lambda, X)$ is invariant under the adjoint action of a subgroup of $GL(n+1)$ that consists of matrices whose off-diagonal entries in the first row and column are zero, one concludes that for any $(n+1)\times(n+1)$ matrix A, that satisfies this property, $Tr\left(\left[\nabla_\lambda, X\right]A\right) = \langle\left[\nabla_\lambda, X\right], A\rangle = 0$. Then (3.20) can be re-written as

$$\begin{aligned}\{M(\lambda,X),M(\mu,X)\}_k &= \left\langle (X^k\nabla_\lambda)_{>0}, [\nabla_\mu, X]\right\rangle - \left\langle [\nabla_\lambda, X], (\nabla_\mu X^k)_{\le 0}\right\rangle\\ &= \left\langle X, \left[(X^k\nabla_\lambda)_{>0}, \nabla_\mu\right] - \left[\nabla_\lambda, (\nabla_\mu X^k)_{\le 0}\right]\right\rangle \ .\end{aligned} \qquad (3.21)$$

Next, since (3.14) defines a skew-symmetric bracket on Rat'_{n+1} and, hence, $\{M(\lambda,X),M(\mu,X)\}_k = \frac{1}{2}\left(\{M(\lambda,X),M(\mu,X)\}_k - \{M(\mu,X),M(\lambda,X)\}_k\right)$, (3.21) implies

$$\begin{aligned}\{M(\lambda,X),M(\mu,X)\}_k &= \tfrac{1}{2}\left\langle X, \left[(X^k\nabla_\lambda)_{>0} - (\nabla_\lambda X^k)_{\le 0}, \nabla_\mu\right] + \left[\nabla_\lambda, (X^k\nabla_\mu)_{>0} - (\nabla_\mu X^k)_{\le 0}\right]\right\rangle\\ &= \tfrac{1}{2}\left\langle X, \left[(X^k\nabla_\lambda + \nabla_\lambda X^k)_{>0}, \nabla_\mu\right] + \left[\nabla_\lambda, (X^k\nabla_\mu + \nabla_\mu X^k)_{>0}\right] - \left[\nabla_\lambda X^k, \nabla_\mu\right] - \left[\nabla_\lambda, \nabla_\mu X^k\right]\right\rangle\\ &= \tfrac{1}{2}\left\langle X, -\left[(X^k\nabla_\lambda + \nabla_\lambda X^k)_{\le 0}, \nabla_\mu\right] - \left[\nabla_\lambda, (X^k\nabla_\mu + \nabla_\mu X^k)_{\le}\right] + \left[X^k\nabla_\lambda, \nabla_\mu\right] + \left[\nabla_\lambda, X^k\nabla_\mu\right]\right\rangle\end{aligned}$$

Taking the average of the last two lines and observing that, due to the Jacobi identity, $\left\langle X, \left[[X^k, \nabla_\lambda], \nabla_\mu\right]\right\rangle = \left\langle X, \left[\nabla_\lambda, , [\nabla_\mu, X^k]\right]\right\rangle = 0$, one obtains (3.18). □

4

In [8], master symmetries were used to establish a multi-Hamiltonian structure of the symmetric finite non-periodic Toda lattice. (For a definition and examples of master symmetries see, e.g., [13].) Namely, a family of vector fields $Y_m, m \ge 1$ was constructed, that satisfies the properties, that, in the context of the non-symmetric Toda lattice, can be described as follows. Let $\mathfrak{L}_Y$ denote the Lie derivative in the direction of the vector field Y, ν_m be the Hamiltonian vector field on

Jac defined by the right-hand side of (2.1) and let functions H_j be defined by (3.17). Then (i) $[Y_l, Y_m] = (l-m)Y_{l+m}$; (ii) $\mathfrak{L}_{Y_l} H_j = (l+j)H_{l+j}$; (iii) $[Y_l, \nu_m] = (m-1)\nu_{m+l}$ and (iv) Lie derivatives $\mathfrak{L}_{Y_l}\{\ ,\ \}_0$ of the Poisson bracket obtained as a restriction of the bracket (2.2) to Jac are compatible Poisson brackets on Jac. For $l = 1, 2$, Y_l was defined via a differential equation of the form

$$\dot{X} = X^{l+1} + [X, B_l(X)] \ , \tag{4.1}$$

where an auxiliary matrix $B_l(X)$ was chosen in such a way, that the right-hand side of (4.1) is tridiagonal (such choice is not unique). For $l > 2$, vector fields Y_l were defined recursively as $Y_l = \frac{1}{l-2}[Y_1, Y_{l-1}]$. Various extensions of the results of [8] can be found in [9].

Using results from the previous section, we shall give explicit formulae for master symmetries Y_l. For all l, they will be described by nonisospectral equations of the form (4.1).

First, we need to modify the results of [5], [6] to describe nonisospectral flows on Jac. Recall that for any $X \in Jac, \lambda \in \mathbb{C}$ there exist uniquely defined vectors $P(\lambda) = (p_i(\lambda))_{i=0}^n, \tilde{P}(\lambda) = (\tilde{p}_i(\lambda))_{i=0}^n$, such that $p_0(\lambda) = \tilde{p}_0(\lambda) = 1$ and

$$XP(\lambda) = \lambda P(\lambda) - p_{n+1}(\lambda)e_n, \quad \tilde{P}(\lambda)^T X = \lambda \tilde{P}(\lambda)^T - \tilde{p}_{n+1}(\lambda)e_n^T \ . \tag{4.2}$$

Necessarily, $p_i(\lambda), \tilde{p}_i(\lambda),\ i = 1, \dots, n+1$ are polynomials of degree i (moreover, $\tilde{p}_i(\lambda)$ are monic). Furthermore, $p_{n+1}(\lambda)$ is equal to and $\tilde{p}_{n+1}(\lambda)$ is a scalar multiple of the characteristic polynomial of X. Consequently,

$$\frac{d}{d\lambda}P(\lambda) = \mathcal{D}P(\lambda), \quad \frac{d}{d\lambda}\tilde{P}(\lambda)^T = \tilde{P}(\lambda)^T\tilde{\mathcal{D}}, \tag{4.3}$$

where $\mathcal{D}$ (resp. $\tilde{\mathcal{D}}$) is a uniquely defined strictly lower (resp. upper) triangular matrix independent of λ. Then a differentiation of equalities (4.2) w.r.t. λ leads to the following relations

$$[X, \mathcal{D}] = \mathbf{1} + e_n v^T, \quad \left[X, \tilde{\mathcal{D}}\right] = -\mathbf{1} + \tilde{v}e_n^T \ , \tag{4.4}$$

where v_r, v_l are vectors that depend on X.

Remark. It is worth mentioning that pairs $(X, -\mathcal{D})$ and $X, \tilde{\mathcal{D}}$ belong to a class matrix pairs (X, Z) satisfying a condition $\operatorname{rank}([X, Z] + \mathbf{1}) = 1$. This class plays an important role in study of classical and quantum solvable models, see, e.g. [24], where it was studied in connection with the Calogero-Moser model, and [16] where it was used to derive a discrete time integrable system for the energies certain solvable quantum models.

For any polynomial $Q(\lambda) = \sum_{k=0}^m Q_k\lambda^k$, consider now a differential equation

$$\dot{X} = Q(X) + \left[X, (Q(X)\tilde{\mathcal{D}})_{\le 0} - (\mathcal{D}Q(X))_{>0}\right] \ , \tag{4.5}$$

Proposition 4.1 *The vector field defined by (4.5) is tangent to Jac. If $X(t)$ evolves according to (4.5), then the evolution of functions $\rho_i = \rho_i(X(t)), \lambda_i = \lambda_i(X(t)),\ i = 0, \dots n$ and $s_j = s_j(X(t)),\ j = 0, \dots$ defined in (2.4), (2.5) is given by equations*

$$\dot{\rho}_i = 0,\ \dot{\lambda}_i = Q(\lambda_i) \ . \tag{4.6}$$

and

$$\dot{s}_j = j\sum_{k=0}^m Q_k s_{k+j-1} \tag{4.7}$$

Proof. To prove the first statement, we have to show that only diagonal and super-diagonal entries of the right hand side of (4.5) are nonzero. First note, that $[X,(\mathcal{D}Q(X))_{>0}]$ is upper triangular. Therefore, the lower triangular part of of the right hand side of (4.5) is equal to $(Q(X)+[X,(Q(X)\tilde{\mathcal{D}})_{\leq 0}])_{<0} = (Q(X))_{<0} + ([X, Q(X)\tilde{\mathcal{D}}])_{<0} = (Q(X))_{<0} + (Q(X)[X,\tilde{\mathcal{D}}])_{<0} = 0$ due to the second equality in (4.4). Similarly, if $A_{>1}$ denotes the part of the matrix A strictly above the super-diagonal, then $(Q(X))_{>1} - [X,(\mathcal{D}Q(X))_{>0}]_{>1} = (Q(X))_{>1} - ([X,\mathcal{D}]Q(X))_{>1} = 0$ by the first equality in (4.4). Since $[X,(Q(X)\tilde{\mathcal{D}})_{\leq 0}]$ is upper Hessenberg, the first statement is proved.

Next, (4.5) implies

$$\dot{X^j} = jX^{j-1}Q(X) + \left[X^j, (Q(X)\tilde{\mathcal{D}})_{\leq 0} - (\mathcal{D}Q(X))_{>0}\right] \ , \tag{4.8}$$

Since $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are, resp., strictly lower and upper triangular, both $(Q(X)\tilde{\mathcal{D}})_{\leq 0}$ and $(\mathcal{D}Q(X))_{>0}$ have zero first row and zero first column. Thus, it follows from (4.8) that $\dot{s}_j = (\dot{X^j}e_0, e_0) = j(X^{j-1}Q(X)e_0, e_0)$ and (4.7) follows. By (2.7), $s_j = \sum_{i=0}^n \lambda_i^m \rho_i(X)$. Then it is easily seen, that (4.6) is consistent with (4.7) and therefore is satisfied by ρ_i, λ_i due to the well-known fact that ρ_i, λ_i are determined uniquely by $s_j (j \geq 0)$. □

Consider now vector fields $\mathcal{V}_l$ $(l = 1, 2, \dots)$ on Rat_{n+1} defined, in coordinates $\lambda_i, q_i = q(\lambda_i)$, by

$$\mathcal{V}_l = \sum_{i=0}^n \lambda_i^{l+1} \frac{\partial}{\partial \lambda_i} \tag{4.9}$$

and let, as before, $H_j = \frac{1}{j}\sum_{l=0}^n \lambda_l^j (j = \pm 1, \pm 2, \dots)$ and $H_0 = \sum_{l=0}^n \ln \lambda_l$ and $\{\ ,\ \}_k$ be the Poisson brackets (3.3), (3.4).

Proposition 4.2 *Vector fields $\mathcal{V}_l$ satisfy the following properties:*

(i) $\mathfrak{L}_{\mathcal{V}_l} H_j = (l+j) H_{l+j}$

(ii) $\mathfrak{L}_{\mathcal{V}_l}\{\ ,\ \}_k = (k-l-1)\{\ ,\ \}_{k+l}$

(iii) $\mathfrak{L}_{\mathcal{V}_l} h_j = (j+l-n)h_{j+l} - \sum_{\beta=1}^l H_\beta h_{j+l-\beta}$

Proof. (i) is obvious. To prove (ii), it suffices to use (3.6), (3.8), (3.8) and an identity $(\mathfrak{L}_{\mathcal{V}}\{\ ,\ \})(f,g) = \mathfrak{L}_{\mathcal{V}}\{f,g\} - \{\mathfrak{L}_{\mathcal{V}}f, g\} - \{f, \mathfrak{L}_{\mathcal{V}}g\}$.

To prove (iii), recall from (3.1), (3.5) that

$$h_j = \sum_{i=0}^n r_i \lambda_i^j = \sum_{i=0}^n \frac{q_i}{p'(\lambda_i)} \lambda_i^j \tag{4.10}$$

Since

$$\begin{aligned} \mathfrak{L}_{\mathcal{V}_l} \ln p'(\lambda_i) &= \textstyle\sum_{\alpha \neq i} \mathfrak{L}_{\mathcal{V}_l} \ln(\lambda_i - \lambda_\alpha) = \sum_{\alpha \neq i} \frac{\lambda_i^{l+1} - \lambda_\alpha^{l+1}}{\lambda_i - \lambda_\alpha} \\ &= \textstyle\sum_{\alpha \neq i} \sum_{\beta=0}^l \lambda_i^\beta \lambda_\alpha^{l-\beta} = (n-l)\lambda_i^l + \sum_{\beta=1}^l \lambda_i^{l-\beta} H_\beta \ , \end{aligned}$$

one obtains from (4.9), (4.10)

$$\mathfrak{L}_{\mathcal{V}_l} h_j = \sum_{i=0}^n r_i((j+l-n)\lambda_i^{j+l} - \sum_{\beta=1}^l \lambda_i^{j+l-\beta} H_\beta) = (j+l-n)h_{j+l} - \sum_{\beta=1}^l H_\beta h_{j+l-\beta} \ .\square$$

We are now ready to prove the following

Theorem 4.3 *Let functions H_j on Jac be defined as in (3.17) and, for $l = 1, 2, \dots$, let $B_l(X) = (X^{l+1}\tilde{\mathcal{D}})_{\leq 0} - (\mathcal{D}X^{l+1})_{>0} + \sum_{\beta=1}^{l} H_\beta (X^{l-\beta})_{\leq 0}$. Then vector fields Y_l on Jac defined by (4.1) satisfy*

(i) $\mathfrak{L}_{Y_l} H_j = (l+j) H_{l+j}$

(ii) $\mathfrak{L}_{Y_l} \{\ ,\ \}_k = (k-l-1)\{\ ,\ \}_{k+l}$, *where* $\{\ ,\ \}_k$ *are compatible Poisson brackets on Jac described by Theorem 3.3.*

Proof. All we need to show is that evolution equations induced by vector fields Y_l on the Weyl function $M(\lambda)$ defined in (2.4) coincide with evolution equations on Rat'_{n+1} induced by vector fields $\mathcal{V}_l$ on Rat_{n+1} via the map $m(\lambda) \to M(\lambda) = \frac{m(\lambda)}{h_0}$. To this end, it suffices to compare equations for Laurent coefficients s_j of $M(\lambda)$. The latter equations do coincide, which drops out immediately from equations (2.6) and Propositions 4.1 and 4.2. □

Combined with Proposition 2.1, Theorem 4.3 allows us to construct master symmetries for the Toda flows on M_I for any I and gives an alternative description of the multi-Hamiltonian structure for integrable lattices TL_I.

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